

# THE INTEGRALS IN GRADSHTEYN AND RYZHIK. PART 7: ELEMENTARY EXAMPLES

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ABSTRACT. The table of Gradshteyn and Ryzhik contains some elementary integrals. Some of them are derived here.

## 1. INTRODUCTION

Elementary mathematics leave the impression that there is marked difference between the two branches of calculus. *Differentiation* is a subject that is systematic: every evaluation is a consequence of a number of rules and some basic examples. However, *integration* is a mixture of art and science. The successful evaluation of an integral depends on the right approach, the right change of variables or a patient search in a table of integrals. In fact, the theory of *indefinite* integrals of elementary functions is complete [3, 4]. Risch's algorithm determines whether a given function has an antiderivative within a given class of functions. However, the theory of *definite* integrals is far from complete and there is no general theory available. The level of complexity in the evaluation of a definite integral is hard to predict as can be seen in

$$(1.1) \quad \int_0^\infty e^{-x} dx = 1, \quad \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}, \quad \text{and} \quad \int_0^\infty e^{-x^3} dx = \Gamma\left(\frac{4}{3}\right).$$

The first integrand has an elementary primitive, the second one is the classical Gaussian integral and the evaluation of the third requires Euler's *gamma function* defined by

$$(1.2) \quad \Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx.$$

The table of integrals [5] contains a large variety of integrals. This paper continues the work initiated in [1, 7, 8, 9, 10, 11] with the objective of providing proofs and context of *all the formulas* in the table [5]. Some of them are truly elementary. In this paper we present a derivation of a small number of them.

## 2. A SIMPLE EXAMPLE

The first evaluation considered here is that of **3.249.6**:

$$(2.1) \quad \int_0^1 (1 - \sqrt{x})^{p-1} dx = \frac{2}{p(p+1)}.$$

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The evaluation is completely elementary. The change of variables  $y = 1 - \sqrt{x}$  produces

$$(2.2) \quad I = -2 \int_0^1 y^p dy + 2 \int_0^1 y^{p-1} dy,$$

and each of these integrals can be evaluated directly to produce the result.

This example can be generalized to consider

$$(2.3) \quad I(a) = \int_0^1 (1 - x^a)^{p-1} dx.$$

The change of variables  $t = x^a$  produces

$$(2.4) \quad I(a) = a^{-1} \int_0^1 t^{1/a-1} (1-t)^{p-1} dt.$$

This integral appears as **3.251.1** and it can be evaluated in terms of the *beta function*

$$(2.5) \quad B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx,$$

as

$$(2.6) \quad I(a) = a^{-1} B(p, a^{-1}).$$

The reader will find in [11] details about this evaluation.

A further generalization is provided in the next lemma.

**Lemma 2.1.** Let  $n \in \mathbb{N}$ ,  $a, b, c \in \mathbb{R}$  with  $bc > 0$ . Define  $u = ac - b^2$ . Then

$$\int_0^1 \frac{a + b\sqrt{x}}{b + c\sqrt{x}} x^{n/2} dx = \frac{2u(-b)^{n+1}}{c^{n+3}} \ln(1 + c/b) + \frac{2u}{c^2} \sum_{j=0}^n \frac{(-1)^j}{n-j+1} \left(\frac{b}{c}\right)^j + \frac{2b}{(n+2)c}.$$

*Proof.* Substitute  $y = b + c\sqrt{x}$  and expand the new term  $(y-b)^n$ .  $\square$

### 3. A GENERALIZATION OF AN ALGEBRAIC EXAMPLE

The evaluation

$$(3.1) \quad \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3}$$

appears as **3.248.4** in [5]. We consider here the generalization

$$(3.2) \quad q(a, b) = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sqrt{b+ax^2}}.$$

We assume that  $a, b > 0$ .

The change of variables  $x = \sqrt{bt}/\sqrt{a}$  yields

$$(3.3) \quad q(a, b) = 2\sqrt{a} \int_0^{\infty} \frac{dt}{(a+bt^2)\sqrt{1+t^2}}$$

where we have used the symmetry of the integrand to write it over  $[0, \infty)$ . The standard trigonometric change of variables  $t = \tan \varphi$  produces

$$(3.4) \quad q(a, b) = 2\sqrt{a} \int_0^{\pi/2} \frac{\cos \varphi d\varphi}{a \cos^2 \varphi + b \sin^2 \varphi}.$$

Finally,  $u = \sin \varphi$ , produces

$$(3.5) \quad q(a, b) = 2\sqrt{a} \int_0^1 \frac{du}{a + (b-a)u^2}.$$

The evaluation of this integral is divided into three cases:

**Case 1.**  $a = b$ . Then we simply get  $q(a, b) = 2/\sqrt{a}$ .

**Case 2.**  $a < b$ . The change of variables  $s = u\sqrt{b-a}/\sqrt{a}$  produces  $(b-a)u^2 = s^2a$ , so that

$$(3.6) \quad q(a, b) = \frac{2}{\sqrt{b-a}} \int_0^c \frac{ds}{1+s^2} = \frac{2}{\sqrt{b-a}} \tan^{-1} c,$$

with  $c = \sqrt{b-a}/\sqrt{a}$ .

**Case 3.**  $a > b$ . Then we write

$$(3.7) \quad q(a, b) = 2\sqrt{a} \int_0^1 \frac{du}{a - (a-b)u^2}.$$

The change of variables  $u = \sqrt{a} s / \sqrt{a-b}$  yields

$$(3.8) \quad q(a, b) = \frac{2}{\sqrt{a-b}} \int_0^c \frac{ds}{1-s^2},$$

where  $c = \sqrt{a-b}/\sqrt{a}$ . The partial fraction decomposition

$$(3.9) \quad \frac{1}{1-s^2} = \frac{1}{2} \left( \frac{1}{1+s} + \frac{1}{1-s} \right)$$

now produces

$$(3.10) \quad q(a, b) = \frac{1}{\sqrt{a-b}} \ln \left( \frac{\sqrt{a} + \sqrt{a-b}}{\sqrt{a} - \sqrt{a-b}} \right).$$

The special case in **3.248.4** corresponds to  $a = 3$  and  $b = 4$ . The value of the integral is  $2 \tan^{-1}(1/\sqrt{3}) = \frac{\pi}{3}$ , as claimed. This generalization has been included as **3.248.6** in the latest edition of [5].

We now consider a generalization of this integral. The proof requires several elementary steps, given first for the convenience of the reader.

Let  $a, b \in \mathbb{R}$  with  $a < b$  and  $n \in \mathbb{N}$ . Introduce the notation

$$(3.11) \quad I = I_n(a, b) := \int_0^\infty \frac{dt}{(a + bt^2)^n \sqrt{1+t^2}}.$$

Then we have:

**Lemma 3.1.** The integral  $I_n(a, b)$  is given by

$$(3.12) \quad I_n(a, b) = \int_0^1 \frac{(1-v^2)^{n-1} dv}{(a + \alpha v^2)^n},$$

with  $\alpha = b - a$ .

*Proof.* The change of variables  $v = t/\sqrt{1+t^2}$  gives the result.  $\square$

The identity

$$(3.13) \quad (1 - v^2)^{n-1} = (1 - v^2)^n + (1 - v^2)^{n-1} \left\{ \frac{1}{\alpha}(a + \alpha v^2) - \frac{a}{\alpha} \right\}$$

produces

$$(3.14) \quad I_n(a, b) = \frac{\alpha}{b} \int_0^1 \frac{(1 - v^2)^n}{(a + \alpha v^2)^n} dv + \frac{1}{b} \int_0^1 \frac{(1 - v^2)^{n-1}}{(a + \alpha v^2)^{n-1}} dv.$$

The evaluation of these integrals requires an intermediate result, that is also of independent interest.

**Lemma 3.2.** Assume  $z \in \mathbb{R}$  and  $n \in \mathbb{N} \cup \{0\}$ . Then

$$(3.15) \quad \int_0^1 \frac{dx}{(1 + z^2 x^2)^{n+1}} = \frac{1}{2^{2n}} \binom{2n}{n} \left( \frac{\tan^{-1} z}{z} + \sum_{k=1}^n \frac{2^{2k}}{2k \binom{2k}{k}} \frac{1}{(1 + z^2)^k} \right).$$

*Proof.* Define

$$(3.16) \quad F_n(z) := \int_0^1 \frac{dx}{(1 + z^2 x^2)^{n+1}} = \frac{1}{z} \int_0^z \frac{dy}{(1 + y^2)^{n+1}}.$$

Take derivatives with respect to  $z$  on both sides of (3.16). The outcome is a system of differential-difference equations

$$(3.17) \quad \begin{aligned} \frac{dF_n(z)}{dz} &= \frac{2(n+1)}{z} F_{n+1}(z) - \frac{2(n+1)}{z} F_n(z) \\ \frac{dF_n}{dz} &= -\frac{1}{z} F_n(z) + \frac{1}{z(1 + z^2)^{n+1}}. \end{aligned}$$

Solving for a purely recursive relation we obtain (after re-indexing  $n \mapsto n-1$ ):

$$(3.18) \quad 2nF_n(z) = (2n-1)F_{n-1}(z) + \frac{1}{(1 + z^2)^n},$$

with the initial condition  $F_0(z) = \frac{1}{z} \tan^{-1} z$ . This recursion is solved using the procedure described in Lemma 2.7 of [1]. This produces the stated expression for  $F_n(z)$ .  $\square$

The next required evaluation is that of the powers of a simple rational function.

**Lemma 3.3.** Let  $a, b, c, d$  be real numbers such that  $cd > 0$ . Then

$$\begin{aligned} \int_0^1 \left( \frac{ax^2 + b}{cx^2 + d} \right)^n dx &= \frac{a^n}{c^n} + \frac{4a^n}{c^n} \sqrt{\frac{d}{c}} \tan^{-1} \sqrt{c/d} \sum_{k=1}^n \left( \frac{bc - ad}{4ad} \right)^k \binom{n}{k} \binom{2k-2}{k-1} \\ &+ \frac{4a^n}{c^n} \sum_{k=1}^n \left( \frac{bc - ad}{4ad} \right)^k \binom{n}{k} \binom{2k-2}{k-1} \sum_{j=1}^{k-1} \frac{2^{2j}}{2^j \binom{2j}{j}} \frac{d^j}{(c+d)^j}. \end{aligned}$$

*Proof.* Start with the partial fraction expansion

$$(3.19) \quad G(x) := \frac{ax^2 + b}{cx^2 + d} = \frac{a}{c} + \frac{bc - ad}{cd} \frac{1}{cx^2/d + 1},$$

and expand  $G(x)^n$  by the binomial theorem. The result follows by using Lemma 3.2.  $\square$

The next result follows by combining the statements of the previous three lemmas.

**Theorem 3.4.** Let  $a, b \in \mathbb{R}^+$  with  $a < b$ . Then

$$\begin{aligned} I_{n+1}(a, b) &:= \int_0^\infty \frac{dt}{(a + bt^2)^{n+1} \sqrt{1+t^2}} \\ &= \frac{1}{a(a-b)^n} \sum_{j=0}^n \binom{n}{j} \left(\frac{-b}{4a}\right)^j \binom{2j}{j} \times \left( \frac{\tan^{-1} \sqrt{b/a-1}}{\sqrt{b/a-1}} + \sum_{k=1}^j \frac{2^{2k}}{2k \binom{2k}{k}} \left(\frac{a}{b}\right)^k \right). \end{aligned}$$

#### 4. SOME INTEGRALS INVOLVING THE EXPONENTIAL FUNCTION

In [5] we find **3.310**:

$$(4.1) \quad \int_0^\infty e^{-px} dx = \frac{1}{p}, \text{ for } p > 0,$$

that is probably the most elementary evaluation in the table. The example **3.311.1**

$$(4.2) \quad \int_0^\infty \frac{dx}{1 + e^{px}} = \frac{\ln 2}{p},$$

can also be evaluated in elementary terms. Observe first that the change of variables  $t = px$ , shows that (4.2) is equivalent to the case  $p = 1$ :

$$(4.3) \quad \int_0^\infty \frac{dx}{1 + e^x} = \ln 2.$$

This can be evaluated by the change of variables  $u = e^x$  that yields

$$(4.4) \quad I = \int_1^\infty \frac{du}{u(1+u)},$$

and this can be integrated by partial fractions to produce the result. The parameter in (4.2) is what we call *fake*, in the sense that the corresponding integral is independent of it. The advantage of such parameter is that it provides flexibility to a formula: differentiating (4.2) with respect to  $p$  produces

$$(4.5) \quad \int_0^\infty \frac{xe^{px} dx}{(1 + e^{px})^2} = \frac{\ln 2}{p^2},$$

$$(4.6) \quad \int_0^\infty \frac{x^2 e^{px} (e^{px} - 1) dx}{(1 + e^{px})^3} = \frac{2 \ln 2}{p^3},$$

$$(4.7) \quad \int_0^\infty \frac{x^3 e^{px} (e^{2px} - 4e^{px} + 1) dx}{(1 + e^{px})^4} = \frac{6 \ln 2}{p^4}.$$

The general integral formula is obtained by differentiating (4.2)  $n$ -times with respect to  $p$  to produce

$$(4.8) \quad \int_0^\infty \left( \frac{\partial}{\partial p} \right)^n \frac{dx}{1 + e^{px}} = (-1)^n \frac{n!}{p^{n+1}} \ln 2.$$

The pattern of the integrand is clear:

$$(4.9) \quad \left( \frac{\partial}{\partial p} \right)^n \frac{1}{1 + e^{px}} = \frac{(-1)^n x^n e^{px}}{(1 + e^{px})^{n+1}} P_n(e^{px}),$$

where  $P_n$  is a polynomial of degree  $n - 1$ . It follows that

$$(4.10) \quad \int_0^\infty \frac{x^n e^{px} P_n(e^{px}) dx}{(1 + e^{px})^{n+1}} = \frac{n! \ln 2}{p^{n+1}}.$$

The change of variables  $t = px$  shows that  $p$  is a fake parameter. The integral is equivalent to

$$(4.11) \quad \int_0^\infty \frac{x^n e^x P_n(e^x) dx}{(1 + e^x)^{n+1}} = n! \ln 2.$$

The first few polynomials in the sequence are given by

$$(4.12) \quad \begin{aligned} P_1(u) &= 1, \\ P_2(u) &= u - 1, \\ P_3(u) &= u^2 - 4u + 1, \\ P_4(u) &= u^3 - 11u^2 + 11u - 1. \end{aligned}$$

**Proposition 4.1.** The polynomials  $P_n(u)$  satisfy the recurrence

$$(4.13) \quad P_{n+1}(u) = (nu - 1)P_n(u) - u(1 + u) \frac{d}{du} P_n(u).$$

*Proof.* The result follows by expanding the relation

$$(4.14) \quad \frac{(-1)^{n+1} x^{n+1} e^{px} P_{n+1}(e^{px})}{(1 + e^{px})^{n+2}} = \frac{\partial}{\partial p} \left( \frac{(-1)^n x^n e^{px} P_n(e^{px})}{(1 + e^{px})^{n+1}} \right).$$

□

Examining the first few values, we observe that

$$(4.15) \quad Q_n(u) := (-1)^n P_n(-u)$$

is a polynomial with positive coefficients. This follows directly from the recurrence

$$(4.16) \quad Q_{n+1}(u) = (nu + 1)Q_n(u) + u(1 - u) \frac{d}{du} Q_n(u).$$

This comes directly from (4.13). The first few values are

$$(4.17) \quad \begin{aligned} Q_1(u) &= 1, \\ Q_2(u) &= u + 1, \\ Q_3(u) &= u^2 + 4u + 1, \\ Q_4(u) &= u^3 + 11u^2 + 11u + 1. \end{aligned}$$

Writing

$$(4.18) \quad Q_n(u) = \sum_{j=0}^{n-1} E_{j,n} u^j,$$

the reader will easily verify the recurrence

$$(4.19) \quad \begin{aligned} E_{0,n+1} &= E_{0,n} \\ E_{j,n+1} &= (n - j + 1)E_{j-1,n} + (j + 1)E_{j,n} \\ E_{n,n+1} &= E_{n,n}. \end{aligned}$$

The numbers  $E_{j,n}$  are called *Eulerian numbers*. They appear in many situations. For example, they provide the coefficients in the series

$$(4.20) \quad \sum_{k=1}^{\infty} k^j x^k = \frac{x}{(1 - x)^{j+1}} \sum_{n=0}^{m-1} E_{j,n} x^n$$

and satisfy the simpler recurrence

$$(4.21) \quad E_{j,n} = nE_{j-1,n} + jE_{j,n-1},$$

that can be derived from (4.19). These numbers have a combinatorial interpretation: they count the number of permutations of  $\{1, 2, \dots, n\}$  having  $j$  permutation ascents. The explicit formula

$$(4.22) \quad E_{j,n} = \sum_{k=0}^{j+1} (-1)^k \binom{n+1}{k} (j+1-k)^n,$$

can be checked from the recurrences. The reader will find more information about these numbers in [6].

## 5. A SIMPLE CHANGE OF VARIABLES

The table [5] contains the example **3.195**:

$$(5.1) \quad \int_0^\infty \frac{(1+x)^{p-1}}{(x+a)^{p+1}} dx = \frac{1-a^{-p}}{p(a-1)}.$$

One must include the restrictions  $a > 0$ ,  $a \neq 1$ ,  $p \neq 0$ . The evaluation is elementary: let

$$(5.2) \quad u = \frac{1+x}{x+a},$$

to obtain

$$(5.3) \quad I = \frac{1}{a-1} \int_{1/a}^1 u^{p-1} du,$$

that gives the stated value. The formula has been supplemented with the value 1 for  $a = 1$  and  $\ln a/(a-1)$  when  $p = 0$  in the last edition of [5].

Differentiating (5.1)  $n$  times with respect to the parameter  $p$  produces

$$\int_0^\infty \frac{(1+x)^{p-1}}{(x+a)^{p+1}} \ln^n \left( \frac{1+x}{x+a} \right) dx = \frac{(-1)^n a^{-p}}{(a-1)^{p+1}} \left[ n! (a^p - 1) - \sum_{k=1}^n \frac{n! (p \ln a)^k}{k!} \right].$$

Naturally, the integral above is just

$$(5.4) \quad \frac{1}{a-1} \int_{1/a}^1 u^{p-1} \ln^n u du$$

and its value can also be obtained by differentiation of (5.3).

The next result presents a generalization of (5.1):

**Lemma 5.1.** Let  $a, b$  be free parameters and  $n \in \mathbb{N}$ . Then

$$\int_0^\infty \frac{(1+x)^b}{(x+a)^{b+n}} dx = (a-1)^{-n} \times \left\{ B(n, b) - \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{a^{-b-k}}{b+k} \right\},$$

where  $B(n, b)$  is Euler's beta function.

*Proof.* Use the change of variables  $u = (1+x)/(a+x)$ , expand in series and then integrate term by term.  $\square$

## 6. ANOTHER EXAMPLE

The integral in **3.268.1** states that

$$(6.1) \quad \int_0^1 \left( \frac{1}{1-x} - \frac{px^{p-1}}{1-x^p} \right) dx = \ln p.$$

To compute it, and to avoid the singularity at  $x = 1$ , we write

$$(6.2) \quad I = \lim_{\epsilon \rightarrow 0} \int_0^{1-\epsilon} \left( \frac{1}{1-x} - \frac{px^{p-1}}{1-x^p} \right) dx.$$

This evaluates as

$$(6.3) \quad I = \lim_{\epsilon \rightarrow 0} -\ln \epsilon + \ln(1 - (1-\epsilon)^p) = \lim_{\epsilon \rightarrow 0} \ln \left( \frac{1 - (1-\epsilon)^p}{\epsilon} \right) = \ln p.$$

## 7. EXAMPLES OF RECURRENCES

Several definite integrals in [5] can be evaluated by producing a recurrence for them. For example, in **3.622.3** we find

$$(7.1) \quad \int_0^{\pi/4} \tan^{2n} x \, dx = (-1)^n \left( \frac{\pi}{4} - \sum_{j=0}^{n-1} \frac{(-1)^{j-1}}{2j-1} \right).$$

To check this identity, define

$$(7.2) \quad I_n = \int_0^{\pi/4} \tan^{2n} x \, dx$$

and integrate by parts to produce

$$(7.3) \quad I_n = -I_{n-1} + \frac{1}{2n-1}.$$

From here we generate the first few values

$$I_0 = \frac{\pi}{4}, I_1 = -\frac{\pi}{4} + 1, I_2 = \frac{\pi}{4} - 1 + \frac{1}{3}, \text{ and } I_3 = -\frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5},$$

and from here one can *guess* the formula (7.1). A proof by induction is easy using (7.3).

A similar argument produces **3.622.4**:

$$(7.4) \quad \int_0^{\pi/4} \tan^{2n+1} x \, dx = \frac{(-1)^{n+1}}{2} \left( \ln 2 - \sum_{k=1}^n \frac{(-1)^k}{k} \right).$$

To establish this, define

$$(7.5) \quad J_n = \int_0^{\pi/4} \tan^{2n+1} x \, dx$$

and integrate by parts to produce

$$(7.6) \quad J_n = -J_{n-1} + \frac{1}{2n}.$$

The value

$$(7.7) \quad J_0 = \int_0^{\pi/4} \tan x \, dx = \frac{\ln 2}{2},$$

and the recurrence (7.6) yield the formula.



## 8. A TRULY ELEMENTARY EXAMPLE

The evaluation of **3.471.1**

$$(8.1) \quad \int_0^u \exp\left(-\frac{b}{x}\right) \frac{dx}{x^2} = \frac{1}{b} \exp\left(-\frac{b}{u}\right),$$

is truly elementary: the change of variables  $t = -b/x$  gives the result.

## 9. COMBINATION OF POLYNOMIALS AND EXPONENTIALS

Integration by parts produces

$$(9.1) \quad \int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx.$$

This appears as **2.321.1** in [5]. Introduce the notation

$$(9.2) \quad I_n(a) := \int x^n e^{ax} dx$$

so that (9.1) states that

$$(9.3) \quad I_n(a) = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}(a).$$

This recurrence is now used to prove

$$(9.4) \quad I_n(a) = n! e^{ax} \sum_{k=0}^n \frac{(-1)^k x^{n-k}}{(n-k)! a^{k+1}}$$

by an easy induction argument. This appears as **2.321.2**. The case  $1 \leq n \leq 4$  appear as **2.322.1**, **2.322.2**, **2.322.3**, **2.322.4**, respectively.

Integrating (9.4) between 0 and  $u$  produces **3.351.1**:

$$(9.5) \quad \int_0^u x^n e^{-ax} dx = \frac{n!}{a^{n+1}} - e^{-au} \sum_{k=0}^n \frac{n!}{k!} \frac{u^k}{a^{n-k+1}}.$$

This sum can be written in terms of the incomplete gamma function. Details will appear in a future publication. Integrating (9.4) from  $u$  to  $\infty$  produces

$$(9.6) \quad \int_u^\infty x^n e^{-ax} dx = e^{-au} \sum_{k=0}^n \frac{n!}{k!} \frac{u^k}{a^{n-k+1}}.$$

This appears as **3.351.2**.

The special case  $n = 1$  of 3.351.1 appears as **3.351.7**. The cases  $n = 2$  and  $n = 3$  appear as **3.351.8** and **3.351.9**, respectively.

## 10. A PERFECT DERIVATIVE

In section **4.212** we find a list of examples that can be evaluated in terms of the exponential integral function, defined by

$$(10.1) \quad \text{Ei}(x) := \int_{-\infty}^x \frac{e^t}{t} dt$$

for  $x < 0$  and by the Cauchy principal value of (10.1) for  $x > 0$ . An exception is **4.212.7**:

$$(10.2) \quad \int_1^e \frac{\ln x dx}{(1 + \ln x)^2} = \frac{e}{2} - 1.$$

This is an elementary integral: the change of variables  $t = 1 + \ln x$  yields

$$(10.3) \quad I = \frac{1}{e} \int_1^2 \frac{(t-1)}{t^2} e^t dt$$

and to evaluate it observe that

$$(10.4) \quad \frac{(t-1)}{t^2} e^t = \frac{d}{dt} \frac{e^t}{t}.$$

The change of variables  $t = \ln x$  in (10.2) yields

$$(10.5) \quad \int_0^1 \frac{t e^t dt}{(1+t)^2} = \frac{e}{2} - 1.$$

This is **3.353.4** in [5].

The previous evaluation can be generalized by introducing a parameter.

**Lemma 10.1.** Let  $\alpha \in \mathbb{R}$ . Then

$$(10.6) \quad \int_1^e \frac{\ln x dx}{(\alpha + \ln x)^{\alpha+1}} = \frac{e}{(\alpha+1)^\alpha} - \frac{1}{\alpha^\alpha}.$$

*Proof.* Substitute  $t = \alpha + \ln x$  and use

$$(10.7) \quad \frac{d}{dt} \frac{e^t}{t^\alpha} = \frac{t-\alpha}{t^{\alpha+1}} e^t.$$

The case  $\alpha = 1$  corresponds to (10.2). □

## 11. INTEGRALS INVOLVING QUADRATIC POLYNOMIALS

There are several evaluation in [5] that involve quadratic polynomials. We assume, for reasons of convergence, that the discriminant  $d = b^2 - ac$  is strictly negative.

We start with

$$(11.1) \quad \int_0^\infty \frac{dx}{ax^2 + 2bx + c} = \frac{1}{\sqrt{ac - b^2}} \cot^{-1} \left( \frac{b}{\sqrt{ac - b^2}} \right).$$

This is evaluated by completing the square and a simple trigonometric substitution:

$$\begin{aligned} \int_0^\infty \frac{dx}{ax^2 + 2bx + c} &= \frac{1}{a} \int_{b/a}^\infty \frac{du}{u^2 - d/a^2} \\ &= \frac{1}{\sqrt{-d}} \int_{b/\sqrt{-d}}^\infty \frac{dv}{v^2 + 1}. \end{aligned}$$

Differentiating (11.1) with respect to  $c$  produces **3.252.1**:

$$\int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left[ \frac{\cot^{-1}(b/\sqrt{ac - b^2})}{\sqrt{ac - b^2}} \right].$$

We now produce a closed-form expression for this integral.

**Lemma 11.1.** Let  $n \in \mathbb{N}$  and  $u := 4(ac - b^2)/ac$ . Assume  $cu > 0$ . Then we have the explicit evaluation

$$(11.2) \quad \int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^n} = \frac{2b}{a(cu)^n} \binom{2n-2}{n-1} \times \left\{ \frac{\sqrt{acu}}{b} \cot^{-1} \left( \frac{2b}{\sqrt{acu}} \right) - \sum_{j=1}^{n-1} \frac{u^j}{j \binom{2j}{j}} \right\}.$$

*Proof.* The case  $n = 1$  was described above:

$$(11.3) \quad h(a, b, c) := \int_0^\infty \frac{dx}{ax^2 + 2bx + c} = \frac{1}{\sqrt{ac - b^2}} \cot^{-1} \left( \frac{1}{\sqrt{ac - b^2}} \right).$$

Now observe that  $h(a^2, abc, b^2) = h(1, b, 1)/ac$ . To establish (11.2), change the parameters sequentially as  $a \mapsto a^2$ ;  $c \mapsto c^2$ ;  $b \mapsto abc$ . In the new format, both sides satisfy the differential-difference equation

$$(11.4) \quad -2nc(1 - b^2)f_{n+1} = \frac{df_n}{dc} + \frac{b}{ac^{2n}}.$$

The identity (11.2) is obtained by reversing the transformations of parameters indicated above.  $\square$

**Corollary 11.2.** Using the notations of Lemma 11.1 we have

$$(11.5) \quad \sum_{j=1}^\infty \frac{u^j}{j \binom{2j}{j}} = \frac{\sqrt{acu}}{b} \cot^{-1} \left( \frac{2b}{\sqrt{acu}} \right).$$

The integral **3.252.2**

$$(11.6) \quad \int_{-\infty}^\infty \frac{dx}{(ax^2 + 2bx + c)^n} = \frac{(2n-3)!! \pi a^{n-1}}{(2n-2)!! (ac - b^2)^{n-1/2}}$$

reduces via  $u = a(x + b/a)/\sqrt{ac - b^2}$  to Wallis' integral

$$(11.7) \quad \int_0^\infty \frac{du}{(u^2 + 1)^n} = \frac{(2n-3)!!}{(2n-2)!!} \frac{\pi}{2},$$

that appears as **3.249.1**. The reader will find in [2] proofs of Wallis' integral. Observe that the evaluation of **3.252.2** is much simpler than the corresponding half-line example presented in Lemma 11.1.

The last example of this type is **3.252.3**:

$$\int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^{n+3/2}} = \frac{(-2)^n}{(2n+1)!!} \frac{\partial^n}{\partial c^n} \left( \frac{1}{\sqrt{c}(\sqrt{ac} + b)} \right).$$

A simple trigonometric substitution gives the case  $n = 0$ :

$$\begin{aligned} \int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^{3/2}} &= \frac{\sqrt{a}}{ac - b^2} \int_{b/\sqrt{-d}}^\infty \frac{du}{(u^2 + 1)^{3/2}} \\ &= \frac{1}{\sqrt{c}(\sqrt{ac} + b)}. \end{aligned}$$

The general case follows by differentiating with respect to  $c$  and observing that

$$\left( \frac{\partial}{\partial c} \right)^j = (-1)^j \frac{(2j+1)!!}{2^j} (ax^2 + bx + c)^{-3/2-j}.$$

We now provide a closed-form expression for (11.8).

**Theorem 11.3.** Let  $a, b, c \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Define  $u = (ac - b^2)/4ac$  and assume  $cu > 0$ . Then

$$\int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^{n+3/2}} = \frac{(cu)^{-n}}{\sqrt{c} \binom{2n}{n} (2n+1)} \left( \frac{1}{\sqrt{ac} + b} - \frac{b}{ac - b^2} \sum_{j=1}^n \binom{2j}{j} u^j \right).$$

*Proof.* Change parameters sequentially as  $a \mapsto a^2$ ;  $c \mapsto c^2$ ;  $b \mapsto abc$ . Then, in the new format both sides satisfy the differential-difference equation

$$(11.8) \quad -(2N(1-b^2)c)f_{N+1} = \frac{df_N}{dc} - \frac{b}{ac^{2N}},$$

where  $N = n + \frac{3}{2}$ . □

## 12. AN ELEMENTARY COMBINATION OF EXPONENTIALS AND RATIONAL FUNCTIONS

The table [5] contains two integrals belonging to the family

$$(12.1) \quad T_j := \int_0^\infty e^{-px}(e^{-x} - 1)^n \frac{dx}{x^j}.$$

Indeed **3.411.19** gives  $T_1$ :

$$(12.2) \quad \int_0^\infty e^{-px}(e^{-x} - 1)^n \frac{dx}{x} = - \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(p + n - k),$$

and **3.411.20** gives  $T_2$ :

$$(12.3) \quad \int_0^\infty e^{-px}(e^{-x} - 1)^n \frac{dx}{x^2} = \sum_{k=0}^n (-1)^k \binom{n}{k} (p + n - k) \ln(p + n - k),$$

The next result presents an explicit evaluation of  $T_j$ .

**Proposition 12.1.** Let  $p$  be a free parameter, and let  $n, j \in \mathbb{N}$  with  $n + p > 0$ . Then

$$(12.4) \quad \int_0^\infty e^{-px}(e^{-x} - 1)^n \frac{dx}{x^j} = \frac{(-1)^j}{(j-1)!} \sum_{k=0}^n (-1)^k (p + n - k)^{j-1} \ln(p + n - k).$$

*Proof.* Start with the observation that

$$(12.5) \quad T_j = - \int T_{j-1}(p) dp + C.$$

Therefore we need to describe the iterative integrals  $f_j(p) = \int f_{j-1}(p) dp$ , with  $f_0(p) = \ln(p + \alpha)$ . This can be found in page 82 of [2] as

$$(12.6) \quad f_j(p) = \frac{1}{j!} (p + \alpha)^j \ln(p + \alpha) - \frac{H_j}{j!} (p + \alpha)^j + C,$$

with  $\alpha = p + n - k$  and  $H_j = 1 + \frac{1}{2} + \dots + \frac{1}{j}$  is the harmonic number. To build back the functions  $T_j$  employ the fact that, for any polynomial  $Q(n, k)$ ,

$$(12.7) \quad \sum_{k=0}^n (-1)^k (-1)^k \binom{n}{k} Q(n, k) \equiv 0.$$

Consequently,

$$(12.8) \quad T_j = C + \frac{(-1)^{j+1}}{j!} \sum_{k=0}^n (-1)^k (p + n - k)^j \ln(p + n - k).$$

The last step is to check that  $C = 0$ . This follows directly from  $T_j \rightarrow 0$  as  $p \rightarrow \infty$ . The assertion is now validated. □

## 13. AN ELEMENTARY LOGARITHMIC INTEGRAL

Entry **4.222.1** states that

$$(13.1) \quad \int_0^\infty \ln \left( \frac{a^2 + x^2}{b^2 + x^2} \right) dx = (a - b)\pi.$$

In order to establish this, we consider the finite integral

$$(13.2) \quad I(m) := \int_0^m \ln \left( \frac{a^2 + x^2}{b^2 + x^2} \right) dx$$

and then let  $m \rightarrow \infty$ .

Integration by parts gives

$$\begin{aligned} \int_0^m \ln(a^2 + x^2) dx &= m \ln(m^2 + a^2) - 2 \int_0^m \frac{x^2 dx}{a^2 + x^2} \\ &= m \ln(m^2 + a^2) - 2m + 2a^2 \int_0^m \frac{dx}{a^2 + x^2} \\ &= m \ln(m^2 + a^2) - 2m + 2a \tan^{-1} \left( \frac{m}{a} \right). \end{aligned}$$

Therefore

$$I(m) = m \ln \left( \frac{m^2 + a^2}{m^2 + b^2} \right) + 2a \tan^{-1} \left( \frac{m}{a} \right) - 2b \tan^{-1} \left( \frac{m}{b} \right).$$

The limit of the logarithmic part is zero and the arctangent part gives  $(a - b)\pi$  as required.

The generalization

$$(13.3) \quad \int_0^\infty \ln \left( \frac{a^s + x^s}{b^s + x^s} \right) dx = (a - b) \frac{\pi}{\sin(\pi/s)}$$

can be established by elementary methods provided we assume the value

$$(13.4) \quad \int_0^\infty \frac{dx}{1 + x^s} = \frac{\pi}{s \sin(\pi/s)}$$

as given. This integral is evaluated in terms of Euler's beta function in [11]. Indeed, integration by parts gives

$$(13.5) \quad \int_0^y \ln(a^s + x^s) dx = y \ln(a^s + y^s) - sy + sa^s \int_0^y \frac{dx}{a^s + x^s},$$

and similarly for the  $b$ -parameter. Combining these evaluations gives

$$\int_0^y \ln \left( \frac{a^s + x^s}{b^s + x^s} \right) dx = y \ln \left( \frac{a^s + y^s}{b^s + y^s} \right) + sa^s \int_0^y \frac{dx}{a^s + x^s} - sb^s \int_0^y \frac{dx}{b^s + x^s}.$$

Upon letting  $y \rightarrow \infty$ , we observe that the logarithmic term vanishes and a scaling reduces the remaining integrals to (13.4).

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